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THE PROBLEM OF DECENTRALIZED CONTROL WITH A DISCONTINUOUS PAYOFF FUNCTION*

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The problem of decentralized control /1/ concerning the meeting of two players who arrive at a fixed place at random instants of time is considered. The players possess partial information about the uncontrolled factors, that is, they only know the instant of their own arrival at the designated site. In the case of the players, the common aim is to wait for one another while, at the same time, spending the smallest possible amount of time waiting. The discontinuity of the decision-making criterion is a special feature of the problem under consideration. The properties of the optimal strategy (OS) of a group /1-5/, that is, of the two players who are meeting, are investigated and a method is proposed for constructing it. An example of the location of an optimal strategy is presented for the case when there is a uniform distribution of the probability of the instants of arrival in an interval.

1. Formulation of the problem.

Two players have arranged to meet at a designated place. However, the instants of time of their arrival, t_1 and t_2, are unknown in advance and are not decided upon by the players. It is only known that these instants randomly take values from R, that their distribution is identical and that it is independently specified by means of a probability measure μ. It is assumed that the measure μ is defined on a Borel σ-algebra of the space R and that it is positive and regular, i.e. for any measurable $A \subset R$ and any $\varepsilon > 0$, an open set Q and a closed set E exist such that $E \subset A \subset Q$ and $\mu(Q \setminus E) < \varepsilon$ /6/.

Each player may stay at the designated place and wait for his partner for as long as he may desire, and we denote the durations for which the players wait at the designated site by $u_1 \geqslant 0$, $u_2 \geqslant 0$. The payoff of the group of two players (of the performing side) is composed of the agreed-upon value of an encounter ρ, if an encounter has taken place, from which the time expended by the two players in waiting is subtracted. Hence, if the players arrive at the designated site at the instants of time t_1 and t_2 and wait for times u_1 and u_2, their payoff is determined using the formula

$$f(t_1, t_2, u_1, u_2) = \begin{cases} \rho - (t_2 - t_1), & \text{if } t_1 \leqslant t_2 \leqslant t_1 + u_1 \\ \rho - (t_1 - t_2), & \text{if } t_2 \leqslant t_1 \leqslant t_2 + u_2 \\ -u_1 = u_2 & \text{otherwise} \end{cases} \tag{1.1}$$

Let us denote the set of all Borel measurable functions $u: R \to [0, +\infty)$ by U. All possible pairs

$$(u_1(\cdot), u_2(\cdot)), \ u_i(\cdot) \in U \ (i = 1, 2) \tag{1.2}$$

are strategies of the performing side (of the two players), where $u_i(t)$ has the meaning of the length of time for which the i-th player has waited if he has arrived at the designated place at the instant of time t. If the partners have chosen a certain strategy (1.2), the expectation value of the payoff is

$$F(u_1(\cdot), u_2(\cdot)) = \int_R \int_R f(t_1, t_2, u_1(t_1), u_2(t_2))\, d\mu(t_1)\, d\mu(t_2) \tag{1.3}$$

The aim of the group of players is to maximize the functional (1.3).

It will be shown that there exists an optimal strategy (OS) for the performing side (the question of the uniqueness of the OS is not considered) which possesses the properties: a) symmetry, that is, the OS has the form $(u_*(\cdot), u_*(\cdot))$ (Lemma 1); b) stepwise behaviour, that is, the function $u_*(\tau) + \tau$ has the form shown in Fig.1 (Lemma 3). A method of calculating the OS when the measure μ and the value of an encounter ρ are specified is also presented (Theorem 1).

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For brevity, we shall henceforth write simply $d\mu$ instead of $d\mu(t)$.

2. The symmetry and step-wise behaviour of the optimal strategy. We will show that there are symmetry strategies among the optimal strategies.

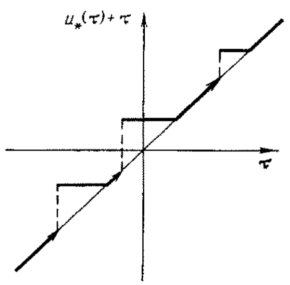

Fig.1

Lemma 1. For any strategy (1.2), there exist functions $v(\cdot)\in U$ such that

$$F(v(\cdot),v(\cdot))\geqslant F(u_1,(\cdot),u_2(\cdot)) \tag{2.1}$$

Proof. The function (1.1) is symmetric, that is, it remains invariant where the indices 1 and 2 are permuted and the functional (1.3) can therefore be preresented in the following manner:

$$F(u_1(\cdot),u_2(\cdot))=F'(u_1(\cdot),u_2(\cdot))+F'(u_2(\cdot),u_1(\cdot))$$

$$F'(u_1(\cdot),u_2(\cdot))=\int_R\Big(\int_{[\tau,\infty)} f(\tau,t,u_1(\tau),u_2(t))\,d\mu\Big)\,d\mu(\tau)$$

Let us put

$$v_*(t)=\min(u_1(t),u_2(t)),\quad v^*(t)=\max(u_1(t),u_2(t))$$

and prove the inequality

$$1/_2\,(F(v^*(\cdot),v^*(\cdot))+F(v_*(\cdot),v_*(\cdot)))\geqslant F(u_1(\cdot),u_2(\cdot)) \tag{2.2}$$

Its left-hand side can be represented in the form

$$\int_R\Big[\Big(\int_{[\tau,\tau+v^*(\tau)]}(\rho-(t-\tau))\,d\mu+\int_{[\tau,\tau+v_*(\tau)]}(\rho-(t-\tau))\,d\mu\Big)-$$
$$(v^*(\tau)\,\mu((\tau+v^*(\tau),\infty))+v_*(\tau)\,\mu((\tau+v_*(\tau),\infty)))-I_v\Big]\,d\mu(\tau)$$

$$I_v=\int_{(\tau+v^*(\tau),\infty)}v^*(t)\,d\mu+\int_{(\tau+v_*(\tau),\infty)}v_*(t)\,d\mu$$

Since, for every fixed $\tau\in R$ either $v^*(\tau)=u_1(\tau)$ and $v_*(\tau)=u_2(\tau)$ or $v^*(\tau)=u_2(\tau)$ and $v_*(\tau)=u_1(\tau)$, the right-hand side of (2.2) only differs in the fact that, instead of the quantity I_v, we shall have

$$I_u=\int_{(\tau+u_1(\tau),\infty)}u_2(t)\,d\mu+\int_{(\tau+u_2(\tau),\infty)}u_1(t)\,d\mu$$

This means that, to prove (2.2), it is sufficient to show that

$$I_v\leqslant I_u,\quad \forall\tau\in R \tag{2.3}$$

For example, let $u_1(\tau)\geqslant u_2(\tau)$. Then, the inequality (2.3) follows from the fact that $\forall t$ $u_1(t)+u_2(t)=v^*(t)+v_*(t)$ and, according to the definition of $v_*(t)$

$$\int_{(\tau+u_2(\tau),\,\tau+u_1(\tau)]}(v_*(t)-u_2(t))\,d\mu\leqslant 0$$

This means that the inequality (2.2) also holds, from which it follows in turn that condition (2.1) is satisfied either for $v(\cdot)=v^*(\cdot)$ or for $v(\cdot)=v_*(\cdot)$ which it was required to prove.

Let us fix a certain function $v(\cdot)\in U$ and consider the function

$$\varphi(t_1,u_1;v(\cdot))=\int_R f(t_1,t_2,u_1,v(t_2))\,d\mu(t_2)$$

Lemma 2. The function $\varphi(t,u;v(\cdot))$ is semicontinuous from above and continuous from the right with respect to u for all $t\in R$ and $v(\cdot)\in U$.

The proof follows from the fact that function (1.1) is semicontinuous from above and continuous from the right with respect to u_1 for all $t_1,t_2,u_2\geqslant 0$ and also from the regularity of the measure μ.

Lemma 3. Among the optimal strategies there exists a symmetric strategy $(v(\cdot),v(\cdot))$ of the form

$$v(t)=\begin{cases}v_i-(t-\theta_i)\ \text{when}\ \tau\in[\theta_i,\theta_i+v_i], & i\in I\\ 0 & \text{otherwise}\end{cases}$$

and, moreover, the intervals $[\theta_i, \theta_i + v_i]$, $i \in I$ do not mutually intersect.

Proof. 1°. For an arbitrary $v(\cdot) \in U$, the function φ can be represented in the form

$$\varphi(\tau, u; v(\cdot)) = \int\limits_{(-\infty, \tau)\setminus L(\tau)} (\rho - (t - \tau))\, d\mu + \int\limits_{L(\tau)} (-u - v(t))\, d\mu + \int\limits_{[\tau, \tau+u]} (\rho - (t - \tau))\, d\mu + \int\limits_{(\tau+u, \infty)} (-u - v(t))\, d\mu \tag{2.4}$$

$$(L(\tau) = \{t \in R \mid t + v(t) < \tau\})$$

Let $\theta \in [\tau, \tau + u]$. Then, by using the representation (2.4), we obtain

$$\varphi(\theta, u - (\theta - \tau); v(\cdot)) = \varphi(\tau, u; v(\cdot)) - Mu + K \tag{2.5}$$

where M and K are independent of u and $M = \mu(L(\theta) - L(\tau)) \geqslant 0$.

Let us now consider a certain symmetric optimal strategy $(u(\cdot), u(\cdot))$ which exists according to Lemma 1. We fix $\tau \in R$ and consider a sequence $\{\tau_j\}$ which converges to τ from the right. Let us assume that, for each $j = 1, 2, \ldots$, there exists $u_j \in \arg\max \varphi(\tau_j, u; u(\cdot))$ such that $u_j + \tau_j \geqslant a$. (Here and subsequently, max is chosen with respect to $u \geqslant 0$ if nothing is stated to the contrary). We shall show that τ then also possesses the same property. For any $j = 1, 2, \ldots$ and $u' \in [0, u_j]$, $\varphi(\tau_j, u_j; u(\cdot)) \geqslant \varphi(\tau_j, u'; u(\cdot))$. Whence, by making use of relationship (2.5), we have

$$\varphi(\tau, u_j + (\tau_j - \tau); u(\cdot)) \geqslant \varphi(\tau, u' + (\tau_j - \tau); u(\cdot))$$

Consequently, for any $u \in (0, u_j$, it is found that $v \geqslant 0$ such that $\tau + v \geqslant a$ and

$$\varphi(\tau, v; u(\cdot)) \geqslant \varphi(\tau, u; u(\cdot)) \tag{2.6}$$

Since the function $\varphi(\tau, u; u(\cdot))$ is continuous from the right with respect to u, (2.6) is also valid when $u = 0$. This means that a $v \in \operatorname{argmax} \varphi(\tau, u; u(\cdot))$ is found for the chosen τ which satisfies the condition $v + \theta \geqslant \tau$.

The assertion which has been proved signifies that, for every instant τ, there exists a minimum $\theta \leqslant \tau$ such that a $v \in \operatorname{argmax} \varphi(\theta, u, u(\cdot))$ is found which satisfies the condition $v + \theta \geqslant \tau$. It follows from inequality (2.6) that, for every $t < \theta$ and $v \in \operatorname{argmax} \varphi(t, u; u(\cdot))$, $v + t < \theta$ is satisfied (otherwise θ would not be the minimum of those such that $v \in \operatorname{argmax} \varphi(\theta, u, u(\cdot))$, $\theta \leqslant \tau \leqslant v + \theta$) exists. This means that there is a set $\{\theta_i \in R, i \in I\}$ for the strategy $(u(\cdot), u(\cdot))$ such that, for any $i \in I$, there exists

$$v_i \in \operatorname{argmax} \varphi(\theta_i, u; u(\cdot)) \tag{2.7}$$

such that

a) $t + v < \theta_i$ if $t < \theta_i$, $v \in \operatorname{argmax} \varphi(t, u; u(\cdot))$

b) $u(t) + t \leqslant \theta_i + v_i$, $t \in [\theta_i, \theta_i + v_i]$.

2°. We shall use the notation

$$v(t) = \begin{cases} v_i - (t - \theta_i), & \text{if } t \in [\theta_i, \theta_i + v_i] \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

Let us show that $F(v(\cdot), v(\cdot)) \geqslant F(u(\cdot), u(\cdot))$. It follows from the definition of the set $L(t)$ that

$$\frac{1}{2} F(u(\cdot), u(\cdot)) = \int\limits_{(-\infty, \infty)} \Big[\int\limits_{(\tau, \tau+u(\tau))} (\rho - (t - \tau))\, d\mu(t) - u(\tau)\mu((\tau + u(\tau), \infty)) - u(\tau)\mu(L(\tau)) \Big]\, d\mu(\tau)$$

Whence, by using the definition of the functional F of the function φ and relationships (2.7), (2.8), we obtain

$$\tfrac{1}{2}F(v(\cdot), v(\cdot)) - \tfrac{1}{2}F(u(\cdot), u(\cdot)) = \sum_{i \in I} \int\limits_{[\theta_i, \theta_i + v_i]} (\varphi(\theta_i, v(\tau) + \tau - \theta_i; u(\cdot)) - \varphi(\theta_i, u(\tau) + \tau - \theta_i; u(\cdot)))\, d\mu(\tau) \tag{2.9}$$

The right-hand side of the inequality (2.9) is non-negative since $v(\tau) + \tau - \theta_i = v_i \in \operatorname{argmax} \varphi(\theta_i, u; u(\cdot))$, $\tau \in [\theta_i, \theta_i + v_i]$.

Hence, the function $v(t) \in U$, defined by relationship (2.8), is optimal and has the form described in the condition of the lemma.

3. Construction of optimal strategies.

Let us consider the function

$$g(\tau, u, u_0) = \int\limits_{[\tau, \tau+v]} (\rho - (t - \tau))\, d\mu - u\mu((-\infty, u) \cup (\tau + u, \infty)) - \int\limits_{(\tau+u, \tau+u_0]} (u_0 - (t - \tau))\, d\mu \tag{3.1}$$

Here, the set $(a, b]$ is assumed to be empty when $a \geqslant b$.

Lemma 4. Let $u_0' < u_0''$. Then, for any τ, $g(\tau, u, u_0') - g(\tau, u, u_0'')$ is a non-negative monotonically increasing function of u and the function $g(\tau, u, u_0)$ is continuous with respect to u_0.

Proof. According to the definition of the function $g(\tau, u, u_0)$

$$g(\tau, u, u_0') - g(\tau, u, u_0'') = \int\limits_{(\tau+u,\, \tau+u_0']} (u_0'' - u_0')\, d\mu + \int\limits_{(\tau+\max(u, u_0'),\, \tau+u_0'']} (u_0'' - (t-\tau))\, d\mu \tag{3.2}$$

The expression on the left-hand side of this equality is non-negative and does not increase monotonically with respect to u since the integrands are non-negative. The continuity of $g(\tau, u, u_0)$ also follows from equality (3.2).

For any τ and u_0 the function $g(\tau, u, u_0)$ is semicontinuous from above and continuous from the right with respect to u. The proof is analogous to the proof of this assertion for $\varphi(\tau, u; v(\cdot))$.

Let us use the notation

$$V_0(\tau) = \{u_0 \geqslant 0 \mid \max g(\tau, u, u_0) = g(\tau, u_0, u_0)\}$$

Theorem 1. A function $u_0(t) \in V_0(t),\ t \in R$ exists such that a strategy $(v(\cdot), v(\cdot))$ gives a maximum to the functional (1.3), where

$$v(\tau) = \max_{t \leqslant \tau} (u_0(t) - \tau + t), \quad \tau \in R \tag{3.3}$$

Proof. According to Lemma 3, a set of pairs $\{(\theta_i, v_i),\ i \in I\}$ exists such that the intervals $[\theta_i, \theta_i + v_i],\ i \in I$ do not mutually intersect and the function

$$v(t) = \begin{cases} v_i + \theta_i - t & \text{when } t \in [\theta_i, \theta_i + v_i],\ i \in I \\ 0 & \text{otherwise} \end{cases}$$

is such that the strategy $(v(\cdot), v(\cdot))$ maximizes the functional (1.3). In the case of such a function $v(t)$, it is necessary to construct a function $u_0(t)$ which satisfies the conditions of the theorem. Since $(v(\cdot), v(\cdot))$ is an optimal strategy, then /1/

$$\varphi(\tau, v(\tau); v(\cdot)) = \max \varphi(\tau, u; v(\cdot)), \ \forall \tau \in R \tag{3.4}$$

When $\tau = \theta_i$ and $u \in [0, v_i]$

$$\varphi(\tau, u; v(\cdot)) = \int\limits_{[\tau, \tau+u]} (\rho - (t-\tau))\, d\mu - \int\limits_{(\tau+u, \tau+v_i]} (v_i - (t-\tau))\, d\mu -$$
$$u\mu((-\infty, \tau) \cup (\tau+u, \infty)) - \int\limits_{(-\infty, \tau) \cup (\tau+v_i, \infty)} v(t)\, d\mu$$

The last term is independent of u and

$$g(\tau, v_i, v_i) \geqslant g(\tau, u, v_i), \ u \in [0, v_i] \tag{3.5}$$

therefore follows from (3.4).

It may be shown in a similar manner that inequality (3.5) also follows from (3.4) in the case when $u > v_i$ since the quantity

$$\int\limits_{(-\infty, \tau) \cup (\tau+u, \infty)} v(t)\, d\mu \tag{3.6}$$

does not decrease with respect to u. Hence, $v_i \in V_0(\theta_i),\ i \in I$.

When $\tau \in [\theta_i, \theta_i + v_i]$, we have $\varphi(\tau, 0, 0) = \max \varphi(\tau, u; v(\cdot))$ and this means that $g(\tau, 0, 0) = \max g(\tau, u, 0)$ since the quantity (3.6) does not decrease with respect to u.

It remains to show that $u_0 \in V_0(\tau),\ u_0 \leqslant v_i - (\tau - \theta_i)$ exists for $\tau \in [\theta_i, \theta_i + v_i]$.

According to the definition of the functions $\varphi(\tau, u; v(\cdot))$ and $g(\tau, u, u_0)$

$$g(\tau, u, v_i - (\tau - \theta_i)) = \varphi(\tau, u; v(\cdot)) - u\mu([\theta_i, \tau]) -$$
$$\int\limits_{(\theta_i+v_i, \infty)} v(t)\, d\mu, \quad u \in [0, v_i - (\tau - \theta_i)]$$

Since $\varphi(\tau, u, v(\cdot))$ reaches a maximum at the point $u = v_i - (\tau - \theta_i)$, it follows from this that a point $\xi_1 \in [0, v_i - (\tau - \theta_i)]$ exists such that

$$g(\tau, \xi_1, v_i - (\tau - \theta_i)) = \max g(\tau, u, v_i - (\tau - \theta_i))$$

According to Lemma 4, there exists $\xi_2 \in [0, \xi_1]$ such that $g(\tau, \xi_2, \xi_1) = \max g(\tau, u, \xi_1)$ and so on.

The sequence $\{\xi_k\}$ does not increase monotonically and is bounded, that is, it has a limit u_0 and, moreover, $u_0 \in [0, v_i - (\tau - \theta_i)]$. For any $u \geqslant 0$, the inequality

$$g(\tau, \xi_{k+1}, \xi_k) \geqslant g(\tau, u, \xi_k), \quad k = 1, 2, \ldots \tag{3.7}$$

is satisfied.

Since the function $g(\tau, u, u_0)$ is continuous with respect to u_0 and continuous from the right with respect to u, by passing to the limit in (3.7), we obtain

$$g(\tau, u_0, u_0) \geqslant g(\tau, u, u_0), \quad u \geqslant 0$$

The latter signifies that $u_0 \in V_0(\tau)$. The function $u_0(\tau)$, $\tau \in R$ which has been constructed is identical with $v(\tau)$ when $\tau \notin [\theta_i, \theta_i + v_i]$ and $u_0(\tau) \leqslant v(\tau)$ when $\tau \in (\theta_i, \theta_i + v_i]$.

Equality (3.3) follows from the definition of $v(t)$ which also completes the proof of the theorem.

Remark 1. The assertion of the theorem also remains true in the case when $\mu(R) < 1$. It is sufficient to define the function $g(\tau, u, u_0)$ as:

$$g(\tau, u, u_0) = \int\limits_{[\tau, \tau+u]} (\rho - (t - \tau))\, d\mu - u(1 - \mu([\tau, \tau + u]) - \int\limits_{(\tau+u, \tau+u_0]} (u_0 - (t - \tau))\, d\mu$$

Let us give an example of the construction of an optimal strategy with the aid of this theorem. Let

$$\mu(A) = \int\limits_A p(t)\, dt, \qquad p(t) = \begin{cases} 1, & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

(a uniform distribution of the probability of arrival at the designated place in the interval $[0, 1]$). Then,

$$g(\tau, u, u_0) = \begin{cases} u(\rho - 1 - u_0), & u \in [0, u_0] \\ \rho u + u^2/2 - u, & u \in (u_0, 1 - \tau] \end{cases}$$

The set of solutions of the equation

$$g(\tau, u_0, u_0) = \max_{u \in [0, 1-\tau]} g(\tau, u, u_0)$$

has the form

$$V_0(\tau) = \begin{cases} \{1 - \tau\}, & \tau \in [0, 2\rho - 1) \\ \{1 - \tau, 0\}, & \tau \in [2\rho - 1, \rho] \\ \{0\}, & \tau \in (\rho, 1] \end{cases}$$

When $\rho \geqslant 1/2$, this means that the unique function which satisfies the conditions of Theorem 1, has the form

$$v(t) = 1 - t, \quad t \in [0, 1]$$

However, if $\rho < 1/2$, then all the functions

$$v_a(t) = \begin{cases} 0, & t \in [0, a] \\ 1 - t, & t \in (a, 1] \end{cases}$$

and also the function $v(t) = 0$, $t \in [0, 1]$ satisfy the conditions of Theoem 1. By calculating the value of the criterion $F(v_a(\cdot), v_a(\cdot))$ and carrying out an optimization with respect to the parameter, we obtain that, when $\rho > 1/3$, the optimal strategy $v(t) = 1 - t, t \in [0, 1]$, when $\rho < 1/3$, the optimal strategy $v(t) = 0, t \in [0, 1]$ and, when $\rho = 1/3$, both strategies are optimal.

Remark 2. If the probability distribution for the time of arrival at the agreed place is not the same in the case of the two players then, generally speaking, the optimal strategy will not possess the properties of symmetry and step-wise behaviour. For example, in the case when

$$\mu_i(A) = \int\limits_A p_i(t)\, dt, \quad i = 1, 2$$

$$p_1(t) = \pi^{-1/2} \exp(-t^2), \quad t \in R$$

$$p_2(t) = \begin{cases} 1/(2b), & \text{if } t \in [-b, b] \\ 0 & \text{otherwise} \end{cases}$$

it can be shown that any optimal strategy is asymmetric and non-step-wise in character for a corresponding choice of the numbers ρ and b.

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ONE SELFMODELLING SOLUTION OF A PROBLEM ON A PLANAR LAMINAR JET*

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The problem of the flow of a laminar jet which does not mix with the fluid surrounding it is treated in the boundary-layer approximation. It is assumed that both fluids are incompressible, that their surface of separation is smooth and that the jet does not break up. A selfmodelling solution (in Mieses variables) of the planar problem is obtained for the special case when the viscosities of the fluids are inversely proportional to their densities.

This problem has been treated previously in the case of a planar /1, 2/ and axially symmetric /3-7/ jet using different versions of the integral method /1, 3, 5, 7/ and also using an asymptotic method /2, 4, 6/ which yields the solution at large distances from the source.

1. The flow domain is shown schematically in Fig.1. Quantities referring to the emitted and external fluids are denoted by means of the indices 1 and 2. The equations of motion in the boundary layer approximation have the form

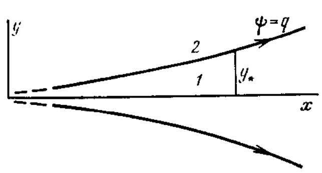

Fig.1

$$u_i\frac{\partial u_i}{\partial x}+v_i\frac{\partial u_i}{\partial y}=\nu_i\frac{\partial^2 u_i}{\partial y^2}, \qquad (1.1)$$
$$\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}=0 \quad (i=1,2)$$

The conditions for the continuity of the velocities and the stresses on the boundary of separation in this approximation as well as the conditions on the axis of the jet and at infinity and the integral relationships expressing the laws of conservation of mass and momentum are represented in the form (only the upper half-plane is considered in view of the symmetry of the problem)

$$y=y_*(x),\ u_1=u_2,\ \mu_1\partial u_1/\partial y=\mu_2\partial u_2/\partial y \qquad (1.2)$$
$$y=0,\ v_1=0,\ \partial u_1/\partial y=0;\ y\to\infty,\ u_2=0$$

$$\int_0^{y_*(x)}\rho_1u_1\,dy=\frac{Q}{2},\quad \int_0^{y_*(x)}\rho_1u_1^2\,dy+\int_{y_*(x)}^{\infty}\rho_2u_2^2\,dy=\frac{J}{2} \qquad (1.3)$$

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